

APPLICATION OF FIBONACCI SERIES, GOLDEN PROPORTIONS AND GOLDEN RECTANGLE TO HUMAN HAND USING STATISTICAL ANALYSIS

SUWATYRA LETCHUMANAN AND RUWAIDIAH IDRIS*

^{1,2} Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia.

*Corresponding author: ruwaidiah@umt.edu.my

Abstract: The Greek letter ϕ (Phi) represents one of the most mysterious numbers (1.618...) known to humankind. Historical approbation for ϕ has led to the monikers “The Golden Number” or “The Divine Proportion”. This simple, but inscrutable number, is inseparably linked to the recursive mathematical sequence which produces Fibonacci numbers. The study of the Fibonacci sequence exists in most aspects of life starting from the leaves of a non-flowering plant, design, paintings, animals, and even human body. Despite its wide-spread prevalence and existence, the Fibonacci series and also the Rule of Golden Proportions have not been widely documented within the human body. The main objective of this study is to prove that the length of the human hand bone is in step with the Fibonacci series to spot the degree of movement and variation for every finger. Victimization of the sample z test with 95% confidence interval, this analysis shows that just one of the four bone length ratios contained the ratio ϕ within the 95% confidence interval and follow the Fibonacci series, that of the little finger metacarpal and proximal phalanx in both hands. The largest variability was seen within the little finger phalangeal relationships and other fingers will follow mathematical relative series. Due to the relationship with the golden number, it will facilitate in monitoring the individual with an injured hand, especially if injured in small fingers throughout a medical aid, or to identify the cause of the problem of physical functioning of the hands or individual fingers. Hence, it should be helpful for the length of the clenched fist to perform in reconstruction or placement of the prosthesis.

Keywords: Fibonacci sequence, golden number, human hand bone, bone length.

Introduction

The concept of Golden ratio or Golden number comes into existence over 24000 years ago as evinced in nature, art and architecture. These ratios appear in various forms in the universe including the geometry of crystals, the magnificence of flowers, artistic work of Renaissance master as well as the proportions on features of the human face and body. Golden ratio also well-known as ‘*Divine Proportion*’ or Phi (ϕ) is a mathematical ratio with its own significant nature (Vorob’ev, 1961). Phi is

solely an irrational number (1.168...) that originated through a recursive mathematical series where a number is added to the number preceding it. The actual value of Phi (ϕ) is closely associated with Golden Rectangle, Golden Angle, Golden Section and Golden Spiral.

Fibonacci numbers are natural numbering system, as they appear in most of these natural phenomena, from the arrangement of leaves till the flower patterns or bract of pine cones. Therefore, Fibonacci numbers exist in the growth of every living

being, including a single cell, wheat, bee hive and human body. The series was discovered in 1202 by an Italian, Leonardo Fibonacci or better known as Fibonacci in Pisa, who derived it from a recursive mathematical sequence. In this sequence, there is a simple addition operation where each number is the sum of two preceding ones. Fibonacci sequences can also be referred to as number sequence of $\{F_n\}_{n=1}^{\infty}$ as defined in the linear equation, $F_n = F_{n-1} + F_{n-2}$ where $F_1 = F_2 = 1$. It can start with 0 and the sequence is as follows (Vorob'ev, 1961):

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

or

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

In addition, the Fibonacci sequence itself is also related to the concept of Golden number. Golden number is an irrational number that is, 1.61833988749..., and it is also represented by the Greek letter Phi (ϕ) or tau (τ) (Debnath, 2011). To avoid mystification, as Golden ratio, Golden number also referred as Phi (ϕ), since both concepts are dealing with the same irrational number (1.618...). The Golden ratio can also be expressed algebraically in terms of a and b with $0 < b < a$ as follows:

$$\frac{a+b}{a} = \frac{a}{b} = \phi,$$

Fibonacci sequence has a recursive relationship of:

$$T_{n+1} = T_n + T_{n-1},$$

with both the second and first series have assisted by:

$$x^2 = x + 1,$$

or

$$x^2 - x - 1 = 0,$$

and the roots are:

$$\phi = x = \frac{1+\sqrt{5}}{2}, \text{ and } \phi = x = \frac{1-\sqrt{5}}{2}.$$

From the theory above or by direct calculation, it can be ascertained that the Golden ratio is an irrational number with its value

$$\phi = x = \frac{1+\sqrt{5}}{2} = 1.61880339887...$$

Golden ratio also can be alluded to as Golden section.

Besides that, Fibonacci sequence is also expressed as a theorem and its proof is as follows (Watson, 2017):

Theorem 1: If each $F_i (i \geq 1)$ is a Fibonacci number, then:

$$\sum_{k=1}^n F_k = F_{k+2} - F_{k+1}$$

Proof:

$$F_1 = F_3 - F_2, \quad F_2 = F_4 - F_3,$$

$$F_3 = F_5 - F_4, \quad F_4 = F_6 - F_5.$$

Apart from that, the Golden rectangle is a rectangle that can be cut into two pieces where one is a square and the other is a rectangle. To be more precise, let ABCD be a rectangle (Refer Figure 1) with its length $CD >$ width AB . There is also a point E on AD segment and a point F on BC segment, thus ABEF is a square with $AE = BF = AB$. While, CDEF is a rectangle. Therefore, ABCD can be said to be a Golden rectangle if its ratio is equal to the ratio of CDEF.

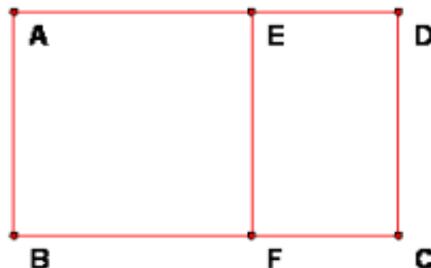


Figure 1: Golden Rectangle (Source: Sruthy, 2012)

The Golden rectangle should also be equal to the Golden ratio, $\frac{\text{length}}{\text{width}} = \text{Golden ratio} = \frac{1+\sqrt{5}}{2}$. In other words, the ratio for the rectangle is 1: 1.618 = 1: ϕ . In this research, we will study the nature of the Golden rectangle and how it is related to the human hand.

Nevertheless, there is another feature of the Fibonacci numbers that illustrates the magnitude of the Golden ratio. Note that if the first number is divided by the next number and the calculation is repeated in several iterations, then it will be converge to an approximation of the Golden ratio or ϕ (1.618...). This shows the properties that correspond to the Fibonacci sequence and all the Golden relationships. Thus, it is clear that one term cannot exist independently of another. Therefore, the terms Fibonacci numbers, Golden ratio or Golden numbers are used interchangeably throughout our study, since basically the three concepts are the same.

Furthermore, the concept of the Golden circle and its relationship to Fibonacci sequence also play an important role in this study. As shown in Figure 2, the Golden spiral is modeled with a set of boxes built using the first ten numbers in the Fibonacci sequence, such that the square has sides with the length of Fibonacci numbers as (1 x 1, 2 x 2, 3 x 3, 5 x 5, 8 x 8, ...).

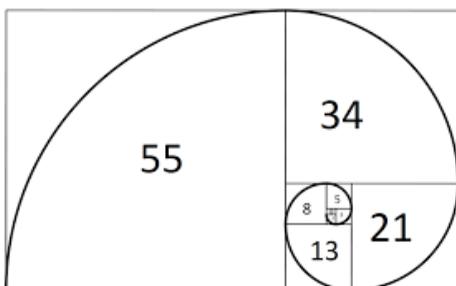


Figure 2: Golden spiral (Source: Dunlap, 1997)

The rectangle surrounding the inner squares is a Golden rectangle that has been divided into a Golden ratio. Figure 2 demonstrates that there are three squares adjacent to a large square which yield the same length as the sides of the large one. One statement said that the Golden rectangle can be divided into smaller squares that carry Fibonacci number dimension called ‘enlarge’ (Dunlap, 1997). Although the mathematical understanding behind the algorithms or Golden spiral is not important, its existence plays a crucial role in the growth and structure of human hands.

In conclusion, the interrelated nature of all the concepts discussed above can be graphically illustrated as in Figure 3.

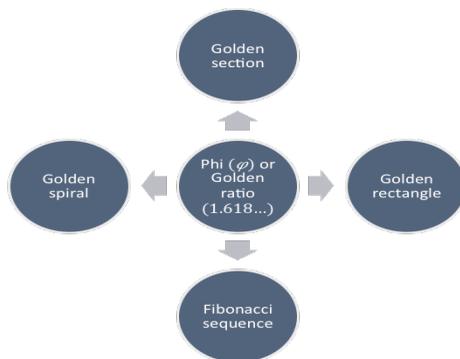


Figure 3: Phi and related concepts (Source: Sruthy, 2012)

The relationship between these four concepts will be discussed further

Fibonacci numbers also play an important role in the structure and movement of human hands. In 1973, a journal entitled “On Adaptability of Man’s Hand” by Littler, an anatomist, who had studied the structure of the hand human and the implications of the golden number. In the journal, Littler (1973) stated that “the structural and functional properties of the human hand can be expressed in mathematical terms that can explain the nature of the internal mechanism”. He also emphasised that the

movement of digits follows the logarithmic or golden circle based on the ratio of gold. In 1998, a team of hand surgeons conducted a scientific study that tested these ideas. The results of their study indicated that finger movements actually follow the path of a golden spiral (Gupta *et al.*, 1998). This relationship is illustrated in Figure 4.

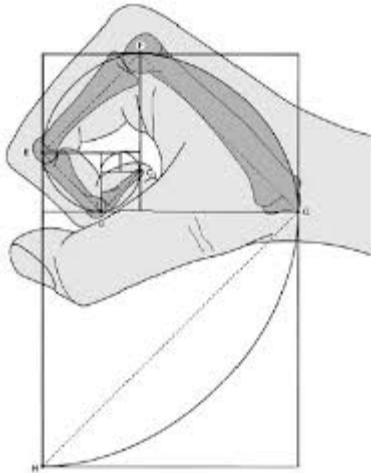


Figure 4: The human hand following the Golden rectangle and the Golden spiral (Source: Andrew *et al.*, 2003)

However, such studies challenge the conclusion that the digits themselves approach the Fibonacci sequence (Andrew *et al.*, 2003). Researchers seek to substantiate or disprove the presumption that if a numerical motion follows the golden spiral, then the length of the digits must also follow the Fibonacci sequence. Unfortunately, due to a lack of statistics and empirical data, the gold circle cannot be easily ascertained and the lack of research that supports that idea.

Materials and Method

Introduction

This section describes the tests performed to get the ratio of each finger. Then, those ratios were compared, and the fingers

were identified that follow the Fibonacci sequence. Figure 5 below briefly describes the flow of the activity.

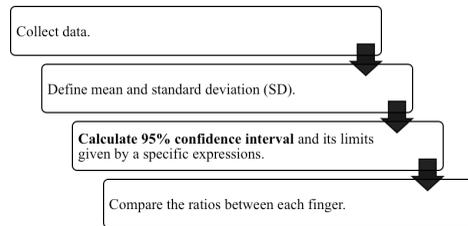


Figure 5: Flow of research activities

Respondents

In this study, 50 respondents, consisting 17 male and 33 female adults were recruited. They were healthy adults with no injuries, trauma or surgery resulting in hand defects. The respondents were anonymous and no clinical information was available other than their age. Their average age is 22.1 (20 – 28 years old).

Measurement and Data collection

We measured the right and left hands of all the respondents whereby they had put their hands in a steady position, without any movement, while measurements were being taken in order to properly represent function. Each data set contains four different phalanges.

We used a scanner as well as CamScanner App to capture the views of hand palms. Respondents laid both their palms on the scanner. If necessary, they were asked to remove jewelry that could interfere with the measurement. Functional finger joints are represented by distal digital crease (distal interphalangeal joint, DIPJ), proximal digital crease (proximal interphalangeal joint, PIPJ), as well as the midpoint between the palmar digital and transverse palmar creases (metacarpophalangeal joint, MCPJ). These creases are illustrated in Figure 6:

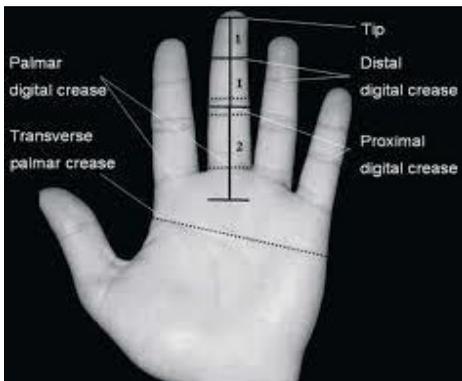


Figure 6: The axis of joints on the palms of the human hand.
(Source: Choo *et al.*, 2012)

The distances between the MCPJ and PIPJ (MCPJ-PIPJ), the PIPJ and DIPJ (PIPJ-DIPJ), as well as the DIPJ and fingertip (DIPJ-Tip) were measured for each finger ray. The ratios of DIPJ-Tip: PIPJ-DIPJ: MCPJ-PIPJ ($P_3 : P_2 : P_1$) were compared to the theoretical ratio of 1:1:2.

Statistical Analysis

The measurements were obtained by using a standard ruler and it was approximately half a millimeter. Each data was made up of alphabet and numbers, so that we are able to differentiate the phalanges. Data were entered and analysed using the Microsoft Excel software.

The length of proximal phalanges is subtracted from the sum of distal and middle phalanges, while the length of metacarpal will be subtracted from the sum of middle and proximal phalanges for each finger. The method is based on the research conducted by Nan Wang *et.al*, 2017 and the same concept was applied in this study. The calculations discussed can be summarised as follows:

$$\text{For each finger } \left\{ \begin{array}{l} (P_1 + P_2) - P_3 \\ (P_3 + P_2) - MC \end{array} \right.$$

with

- P_1 = Length of Distal Phalanges
- P_3 = Length of Proximal Phalanges
- P_2 = Length of Middle Phalanges
- MC = Length of Metacarpal

The calculated values will be included in a table. The length of the adjacent finger was also calculated for comparison with Fibonacci ratios. The 95% confidence interval is given by the following expressions:

$$\bar{x} \pm Z \frac{s}{\sqrt{n}}$$

where

- \bar{x} = Mean length of two phalanges x omits
- s = Standard deviation x
- n = Sample size x
- Z = Selected value from Z distribution table x

The length ratio of the middle and the proximal phalanges ($P_2 : P_3$), ratio of distal phalanges length to middle phalanges ($P_2 : P_3$), and the length ratio of the metacarpal and the proximal phalanges ($MC : P_3$) were calculated separately for each finger.

$$P_2 : P_3 = \frac{P_2}{P_3}, \quad P_1 : P_2 = \frac{P_1}{P_2}, \quad MC : P_3 = \frac{MC}{P_3}$$

95% confidence interval with upper and lower limits would be calculated for each calculation and the calculated values included in a table. Finally, the calculated ratio was compared to calculate the order of the Fibonacci series.

Results and Discussion

Results

The length measurements for both the left and right hands of each respondent were taken in PA (posteroanterior) positioning. The respondent was seated alongside a table and the hand was placed, either palm down on the scanner or palm up towards the CamScanner while ensuring the fingers are of equal distance apart.

95% confidence interval of the difference between the proximal phalanges and the sum of middle and distal phalanges as well as the difference between the metacarpal and the sum of middle and proximal phalanges are as shown in Table 1.

Table 1: 95% confidence interval of right hand.

	95% confidence interval	$(P1+ P2) - P3$	$(P2+ P3) - MC$
Index finger	Lower limit	-0.70	-1.07
	Upper limit	-0.42	-0.63
Middle finger	Lower limit	-0.90	0.07
	Upper limit	-0.57	0.51
Third finger	Lower limit	-0.88	0.12
	Upper limit	-0.57	0.59
Little finger	Lower limit	-0.43	-1.06
	Upper limit	-0.23	-0.60

Table 2 shows 95% confidence intervals for the difference between the proximal and the sum of middle and distal phalanges, as well as the difference between length of metacarpal and the sum of the middle and proximal phalanges of the left hand. However, both tables show that zero is not contained in the 95% confidence interval for any of the relationships in any of the fingers.

Table 2: 95% confidence interval of left hand.

	95% confidence interval	$(P1+ P2) - P3$	$(P2+ P3) - MC$
Index finger	Lower limit	-0.70	-1.11
	Upper limit	-0.37	-0.68
Middle finger	Lower limit	-0.76	-0.15
	Upper limit	-0.40	0.27
Third finger	Lower limit	-0.49	-0.50
	Upper limit	-0.15	-0.06
Little finger	Lower limit	-0.32	-1.19
	Upper limit	-0.07	-0.78

The 95% confidence interval for ratio of the adjacent phalanges is as shown in Table 3 for the right hand, while in Table 4 for the left hand.

Table 3: 95% confidence interval for the ratio of adjacent phalanges for the right hand

	95% confidence interval	P_2/P_1	P_3/P_2	MC/P_3
Index finger	Lower limit	1.576	2.357	1.524
	Upper limit	1.595	2.505	1.591
Middle finger	Lower limit	1.160	2.106	1.380
	Upper limit	1.174	2.137	1.461

Third finger	Lower limit	1.070	2.212	1.335
	Upper limit	1.165	2.231	1.429
Little finger	Lower limit	1.015	2.185	1.600
	Upper limit	1.132	2.353	1.610

Table 4: 95% confidence interval for the ratio of adjacent phalanges for the left hand

95% confidence interval		P_2/P_1	P_3/P_2	MC/P_3
Index finger	Lower limit	1.831	2.222	1.733
	Upper limit	1.914	2.228	1.774
Middle finger	Lower limit	1.183	1.998	1.447
	Upper limit	1.195	2.117	1.534
Third finger	Lower limit	1.071	2.003	1.509
	Upper limit	1.176	2.135	1.513
Little finger	Lower limit	1.006	2.177	1.600
	Upper limit	1.107	2.331	1.691

From this study with 50 respondents, it was found that in both human hands, the ratio of metacarpal and proximal phalanges in small fingers only has a Phi ratio (ϕ) of Fibonacci sequence with 95% confidence interval which is 1.600 ± 1.610 for the right hand, while 1.600 ± 1.610 for the left hand. In other words, 11 confidence intervals for the other fingers do not contain the ratio of Phi (ϕ).

In addition, the small finger shows a greater difference between the upper and lower limits of the ratio. The differences are within 0.1-0.2 for small fingers, while the differences for all the subdivisions including index, middle and third fingers are within 0.04-0.08 range.

Discussion

Many studies have shown that the Golden ratio appears in proportion to the size of the features on the human face as well as the human body (Livio, 2013). On the contrary, some researchers believe that their estimation on human anatomy cannot be proven mathematically (Kal'enine *et al.*, 2013). In this study, we have worked to prove Littler's (1973) hypothesis that the ratio of the human hand with distal, middle and proximal phalanges in each finger follows the Fibonacci sequence. Littler (1973) also described the function of the length of metacarpal and other phalanges as well as their movement path (flexor and extensor) intertwined with the Golden spiral. Thus, the length of the phalanges and metacarpal should follow the Fibonacci relationship, so that it can associate with a Golden rectangle as illustrated in Figure 7.

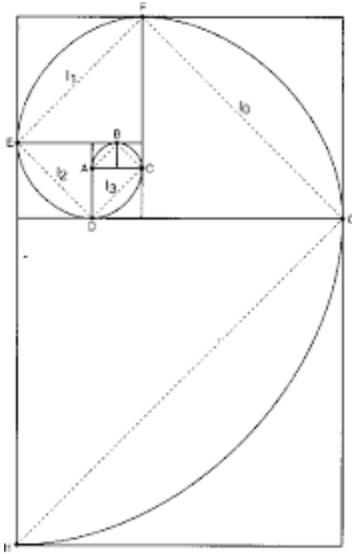


Figure 7: The relationship between the Fibonacci sequence, the Golden rectangle and the Golden spiral.
(Source: Andrew *et al.*, 2003)

In addition, for the anatomy of the human finger bone to follow the Fibonacci sequence, the total length of distal and middle phalanges should be equal to the proximal phalange length of the same finger. Likewise, the sum of the middle and proximal phalanges must be equal to the length of the metacarpal. Based on this statement, Hamilton and Dunsmuir (2002) conducted a study with 30 respondents and a sampling t test to prove that the $p3:p2:p1$ ratio approaches 1:1:2 for small finger while other fingers approach the ratio of 1:1.3:2.3 (Hamilton *et al.*, 2002). Thus, the data collected by Hamilton (2002) support Littler's (1973) early suggestion of the center of rotation of the joints of the hand approaching the dimension of the Golden spiral. Moreover, Hamilton's data also show that the ratio of length of index, middle and third fingers follow Lucas order, while small finger follows Fibonacci sequence (Hamilton & Dunsmuir 2002). The Lucas sequence is not discussed in this study.

This study was conducted based on the Hamilton and Dunsmuir studies (2002). The findings of this study and the z sampling test contradict such relationships as not all lengths of phalanges of a hand follow Fibonacci sequence. When subtracting the length of proximal phalange from the sum of distal and middle phalanges, the zeros are not contained within the 95% confidence interval for the fingers as well as for the difference of metacarpal from the sum of proximal and middle phalanges as shown in Table 1 and Table 2.

Other than the distribution of metacarpal with proximal phalange for small fingers, the Phi (ϕ) ratio was not contained within the 95% confidence interval. The largest intervals of ratio length of the phalanges were calculated on small fingers. The standard deviation of phalanges on small fingers is much larger than for the other fingers. It will show more variation if there is a development of the postaxial skeleton and its implication for its function cannot be determined in this study. However, it is known that the anatomy of the metacarpal joint of the small finger differs from that of the other fingers. These anatomical differences have shown the center of rotation of the metacarpal joint on the small finger with extension and flexion of the finger. Thus, the carpometacarpal of the small finger has a greater degree of movement and variability than the other fingers.

At first glance, it is difficult to understand the reason why the phalanges do not follow the Fibonacci relationship, yet their movement can form a Golden spiral. However, with further studies of human anatomy of the finger, it is clear that the length of phalanges does not play a role in the center of the finger rotation. Thus, we can conclude that the distance from the metacarpal to the center of the metacarpal rotation as well as to the center of the joint between proximal and distal phalanges should produce the length following the Fibonacci relation as illustrated in Figure 4.

Conclusion

Fibonacci numbers hold ascending directions on the structure and movement of human hands. Anatomy expert Littler (1973) has studied the mysteries of human hand and its implications with the Golden number. He has explored the relationship between Fibonacci sequences and human finger movements. By making reference to Littler (1973), and Hamilton and Dunsmuir (2002) were able to succeed in their study of human hands and to prove Littler's statement. Since this study is based on the Hamilton research, we can also prove that the little finger of the human hand follows the Fibonacci sequence. It was demonstrated that 76% out of 50 little fingers of the respondents have provided length measurement data that follows the initial values of a Fibonacci sequence of 0,1,1,2,... represented by $y, y, 2y, \dots$ in Figure 8. In conjunction with the measurement, ratios between the length of Metacarpal (MC) and Proximal phalanges (P_3) approximately converges to Golden Ratio (1.618...) as shown in Table 3 and 4. Thus, with this study we were able to identify a larger level of motion and variability for the small finger compared to other fingers.

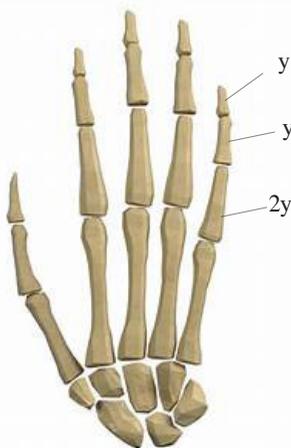


Figure 8: Phalangeal lengths for little finger and ratio of the distances was .
(Source: Hamilton *et al.*, 2002)

Therefore, it can assist in monitoring individuals who have suffered hand injuries during the therapy process and identifying the causes of hand physical functioning problems. Based on this research, it is recommended that instead of using a photocopier machine or application such as CamScanner to get a palm view, hand radiographic imaging will provide more accurate measurements. The length can be calculated more precisely, as we can identify the sequence followed by the other fingers such as the index, middle and ring fingers. In addition, this study could also be extended by examining whether age range affects the Fibonacci relationship by taking the phalangeal lengths of the elderly or children.

Acknowledgements

First, I would like to thank God for because of His bounty and grace I was able to complete this research successfully despite the many obstacles. On this occasion. I would like to express my sincere thanks to Dr. Ruwaidiah Binti Idris, my supervisor for her patience, support, advice and guidance which have greatly contributed to the success of this study. All this help, passion, strategy and wisdom have taught me a lot about being a good student. Not to forget Dr. Azwani Binti Alias, who as my mentor was willing to share her experiences to with me throughout this course. All the experiences I had will never be forgotten. Thank you to all the respondents, students of Universiti Malaysia Terengganu for providing full support which helped facilitate the process of obtaining data. I would also like to thank my loving ~~able~~ parents, Mr. Letchumanan Malai Servai and Mrs. Rajeswary Muthusamy, who always gave me the love, encouragement, prayers, reminders and guidance I needed. Indeed, all the sacrifices they have made are greatly appreciated and will be remembered for the rest of my life. Lastly, thank you to all those

involved directly and indirectly in providing helpful suggestions and assistance in the preparation of this study.

References

- Andrew E. Park, John J. Fernandez, Karl Schmedders, & Mark S. Cohen. (2003). "The Fibonacci Sequence: Relationship to the Human Hand". *The Journal of Hand Surgery* 28(1), 157 –160. non italics please
- Choo. K. W., Quah W., Chang G., & Chan J. Y. (2012). *Functional hand proportion is approximated by the Fibonacci series. Folia Morphologica*, 71(3), 148 – 153. non italics please
- Debnath, Lokenath. (2011). "A Short History of the Fibonacci and Golden Numbers with Their Applications". *International Journal of Mathematical Education in Science and Technology*, 42(3), 337-367. non italics please
- Dunlap, R. A. (1997). *The Golden Ratio and Fibonacci Numbers*, 1st edition. World Scientific Publishing Company, Singapore.
- Hamilton, R., & Dunsmuir R. A. (2002). *Radiographic assessment of the relative lengths of the bones of the fingers of the human hand. The Journal of Hand Surgery: British & European*, 27, 546 – 548. non italics please
- Kal'entine S., Cheam C., Izard V., & Gentaz E., (2013). "Adults and 5-year-old children draw rectangles and triangles around a prototype but not in the golden ratio," *British Journal of Psychology* 104(3), 400 – 412. non italics please.
- Littler, J. W. (1973). *On the Adaptability of Man's Hand (With Reference to the Equiangular Curve)*. *Hand*, 5(3), 187 – 191. non italics please
- Livio, Mario. (2002). "The Golden Ratio: The Story of Phi, The World's Most Astonishing Number." 1st edition. Broadway Books, New York.
- Nan Wang, Jie Ma, Dan Jin, & Bin Yu, (2017). "A Special Golden Curve in Human Upper Limbs' Length Proportion: A Functional Partition Which Is Different from Anatomy". *BioMed Research International* volume 2017, Article ID 4158561, 6 Pages. <https://doi.org/10.1155/2017/4158561> non italics please non italics please non italics please
- Sruthy. M. (2012). *Golden Ratio in Human Anatomy*. Summer Vacation Research Work, Government College, Chittur Palakkad, Kerala.
- Vorob'ev, N. N., (1983). *Fibonacci Numbers*, 1st edition. Dover Publications, America.
- [Translated from the Russian by Halina Moss; translation editor Ian N
- NN Vorob'ev - 1961 - ... Blaisdell Publishing Co.]
- Watson, A. R. (2017). *The Golden Relationships: An Exploration of Fibonacci Numbers and Phi*. Masters Theses, Duke University.